LOGIC, TRUTH AND PROBABILITY

Gunn Quznetsov
quznets@geocities.com

August 25, 1998

Abstract

The propositional logic is generalized on the real numbers field. The logical analog of the Bernoulli independent tests scheme is constructed. The variant of the nonstandard analysis is adopted for the definition of the logical function, which has all properties of the classical probability function. The logical analog of the Large Number Law is deduced from properties of this function.

Contents

1 INTRODUCTION **2**

2 THE NONSTANDARD NUMBERS **2**

3 THE CLASSICAL LOGIC. **5**

4 B-FUNCTIONS **8**

5 THE INDEPENDENT TESTS **9**

6 THE PROBABILITY FUNCTION **11**

7 RESUME **14**

8 APPENDIX I. Consistency **14**

1 INTRODUCTION

There is the evident nigh affinity between the classical probability function and the Boolean function of the classical propositional logic. These functions are differed by the range of value, only. That is if the range of values of the Boolean function shall be expanded from the two-elements set $\{0;1\}$ to the segment $[0;1]$ of the real numeric axis then the logical analog of the Bernoulli Large Number Law [1] can be deduced from the logical axioms. And if the range of values of such function shall be expanded to the segment of some suitable variant of the hyperreal numeric axis then this theorem shall insert some new nonstandard purport for the notion "truth". These topics is considered in this article.

2 THE NONSTANDARD NUMBERS

Let us consider the set $\mathbf{N}$ of natural numbers.

Definition 2.1: The *n-part-set* $\mathbf{S}$ of $\mathbf{N}$ is defined recursively as follows:

1) $\mathbf{S}_1 = \{1\}$;

2) $\mathbf{S}_{(n+1)} = \mathbf{S}_n \cup \{n+1\}$.

Definition 2.2: If $\mathbf{S}_n$ is the n-part-set of $\mathbf{N}$ and $\mathbf{A} \subseteq \mathbf{N}$ then $\|\mathbf{A} \cap \mathbf{S}_n\|$ is the quantity elements of the set $\mathbf{A} \cap \mathbf{S}_n$, and if

$$\varpi_n(\mathbf{A}) = \frac{\|\mathbf{A} \cap \mathbf{S}_n\|}{n},$$

then $\varpi_n(\mathbf{A})$ is *the frequency* of the set $\mathbf{A}$ on the n-part-set $\mathbf{S}_n$.

Theorem 2.1:

1) $\varpi_n(\mathbf{N}) = 1$;

2) $\varpi_n(\emptyset) = 0$;

3) $\varpi_n(\mathbf{A}) + \varpi_n(\mathbf{N} - \mathbf{A}) = 1$;

4) $\varpi_n(\mathbf{A} \cap \mathbf{B}) + \varpi_n(\mathbf{A} \cap (\mathbf{N} - \mathbf{B})) = \varpi_n(\mathbf{A})$.

Definition 2.3: If "lim" is the Cauchy-Weierstrass "limit" then let us denote:

$$\mathbf{\Phi ix} = \left\{ \mathbf{A} \subseteq \mathbf{N} | \lim_{n \to \infty} \varpi_n(\mathbf{A}) = 1 \right\}.$$

Theorem 2.2: $\mathbf{\Phi ix}$ is the filter [2], i.e.:

1) $\mathbf{N} \in \mathbf{\Phi ix}$,
2) $\emptyset \notin \mathbf{\Phi ix}$,
3) if $\mathbf{A} \in \mathbf{\Phi ix}$ and $\mathbf{B} \in \mathbf{\Phi ix}$ then $(\mathbf{A} \cap \mathbf{B}) \in \mathbf{\Phi ix}$;
4) if $\mathbf{A} \in \mathbf{\Phi ix}$ and $\mathbf{A} \subseteq \mathbf{B}$ then $\mathbf{B} \in \mathbf{\Phi ix}$.

In the following text we shall adopt to our topics the definitions and the proofs of the Robinson Nonstandard Analysis [3]:

Definition 2.4: The sequences of the real numbers $\langle r_n \rangle$ and $\langle s_n \rangle$ are *Q-equivalent* (denote: $\langle r_n \rangle \sim \langle s_n \rangle$) if

$$\{n \in \mathbf{N} | r_n = s_n\} \in \mathbf{\Phi ix}.$$

Theorem 2.3: If $\mathbf{r}$,$\mathbf{s}$,$\mathbf{u}$ are the sequences of the real numbers then

1) $\mathbf{r} \sim \mathbf{r}$,
2) if $\mathbf{r} \sim \mathbf{s}$ then $\mathbf{s} \sim \mathbf{r}$;
3) if $\mathbf{r} \sim \mathbf{s}$ and $\mathbf{s} \sim \mathbf{u}$ then $\mathbf{r} \sim \mathbf{u}$.

Definition 2.5: *The Q-number* is the set of the Q-equivalent sequences of the real numbers, i.e. if $\widetilde{a}$ is the Q-number and $\mathbf{r} \in \widetilde{a}$ and $\mathbf{s} \in \widetilde{a}$, then $\mathbf{r} \sim \mathbf{s}$; and if $\mathbf{r} \in \widetilde{a}$ and $\mathbf{r} \sim \mathbf{s}$ then $\mathbf{s} \in \widetilde{a}$.

Definition 2.6: The Q-number $\widetilde{a}$ is *the standard Q-number* a if a is some real number and the sequence $\langle r_n \rangle$ exists, for which: $\langle r_n \rangle \in \widetilde{a}$ and

$$\{n \in \mathbf{N} | r_n = a\} \in \mathbf{\Phi ix}.$$

Definition 2.7: The Q-numbers $\widetilde{a}$ and $\widetilde{b}$ are *the equal Q-numbers* (denote: $\widetilde{a} = \widetilde{b}$) if a $\widetilde{a} \subseteq \widetilde{b}$ and $\widetilde{b} \subseteq \widetilde{a}$.

Theorem 2.4: Let $\mathfrak{f}(x, y, z)$ be a function, which has got the domain in $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$, has got the range of values in $\mathbf{R}$ ($\mathbf{R}$ is the real numbers set).

Let $\langle y_{1,n} \rangle$, $\langle y_{2,n} \rangle$, $\langle y_{3,n} \rangle$, $\langle z_{1,n} \rangle$, $\langle z_{2,n} \rangle$, $\langle z_{3,n} \rangle$ be any sequences of real numbers.

In this case if $\langle z_{i,n} \rangle \sim \langle y_{i,n} \rangle$ then $\langle \mathfrak{f}(y_{1,n}, y_{2,n}, y_{3,n}) \rangle \sim \langle \mathfrak{f}(z_{1,n}, z_{2,n}, z_{3,n}) \rangle$.

Definition 2.8: Let us denote: $Q\mathbf{R}$ is the set of the Q-numbers.

Definition 2.9: The function $\widetilde{\mathfrak{f}}$, which has got the domain in $Q\mathbf{R} \times Q\mathbf{R} \times Q\mathbf{R}$, has got the range of values in $Q\mathbf{R}$, is *the Q-extension of the function* $\mathfrak{f}$, which has got the domain in $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$, has got the range of values in $\mathbf{R}$, if the following condition is accomplished:

Let $\langle x_n \rangle$,$\langle y_n \rangle$,$\langle z_n \rangle$ be any sequences of real numbers. In this case: if

$\langle x_n \rangle \in \widetilde{x}$, $\langle y_n \rangle \in \widetilde{y}$, $\langle z_n \rangle \in \widetilde{z}$, $\widetilde{u} = \widetilde{\mathfrak{f}}\,(\widetilde{x}, \widetilde{y}, \widetilde{z})$,

then

$\langle \mathfrak{f}\,(x_n, y_n, z_n) \rangle \in \widetilde{u}$.

Theorem 2.5: For all functions $\mathfrak{f}$, which have the domain in $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$, have the range of values in $\mathbf{R}$, and for all real numbers a, b, c, d: if $\widetilde{\mathfrak{f}}$ is the Q-extension of $\mathfrak{f}$; $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, $\widetilde{d}$ are standard Q-numbers a, b, c, d, then:

if $d = \mathfrak{f}(a, b, c)$ then $\widetilde{d} = \widetilde{\mathfrak{f}}(\widetilde{a}, \widetilde{b}, \widetilde{c})$ and vice versa.

By this Theorem: if $\widetilde{\mathfrak{f}}$ is the Q-extension of the function $\mathfrak{f}$ then the expression "$\widetilde{\mathfrak{f}}(\widetilde{x}, \widetilde{y}, \widetilde{z})$" will be denoted as "$\mathfrak{f}(\widetilde{x}, \widetilde{y}, \widetilde{z})$" and if $\widetilde{u}$ is the standard Q-number then the expression "$\widetilde{u}$" will be denoted as "u".

Theorem 2.6: If for all real numbers a, b, c:

$$\varphi(a, b, c) = \psi(a, b, c)$$

then for all Q-numbers $\widetilde{x}$, $\widetilde{y}$, $\widetilde{z}$:

$$\varphi(\widetilde{x}, \widetilde{y}, \widetilde{z}) = \psi(\widetilde{x}, \widetilde{y}, \widetilde{z}).$$

Theorem 2.7: If for all real numbers a, b, c:

$$\mathfrak{f}\,(a, \varphi(b, c)) = \psi(a, b, c)$$

then for all Q-numbers $\widetilde{x}$, $\widetilde{y}$, $\widetilde{z}$:

$$\mathfrak{f}\,(\widetilde{x}, \varphi(\widetilde{y}, \widetilde{z})) = \psi(\widetilde{x}, \widetilde{y}, \widetilde{z}).$$

Consequences from Theorems 2.6 and 2.7: [4]: For all Q-numbers $\widetilde{x}$, $\widetilde{y}$, $\widetilde{z}$:

Φ1: $(\widetilde{x} + \widetilde{y}) = (\widetilde{y} + \widetilde{x})$,

Φ2: $(\widetilde{x} + (\widetilde{y} + \widetilde{z})) = ((\widetilde{x} + \widetilde{y}) + \widetilde{z})$,

Φ3: $(\widetilde{x} + 0) = \widetilde{x}$,

Φ5: $(\widetilde{x} \cdot \widetilde{y}) = (\widetilde{y} \cdot \widetilde{x})$,

Φ6: $(\widetilde{x} \cdot (\widetilde{y} \cdot \widetilde{z})) = ((\widetilde{x} \cdot \widetilde{y}) \cdot \widetilde{z})$,

Φ7: $(\widetilde{x} \cdot 1) = \widetilde{x}$,

Φ10: $(\widetilde{x} \cdot (\widetilde{y} + \widetilde{z})) = ((\widetilde{x} \cdot \widetilde{y}) + (\widetilde{x} \cdot \widetilde{z}))$.

Theorem 2.8: Φ4: For every Q-number $\widetilde{x}$ the Q-number $\widetilde{y}$ exists, for which:

$(\widetilde{x} + \widetilde{y}) = 0$.

Theorem 2.9: $\mathbf{\Phi 9}$: There is not that $0 = 1$.

Definition 2.10: The Q-number $\widetilde{x}$ is *Q-less* than the Q-number $\widetilde{y}$ (denote: $\widetilde{x} < \widetilde{y}$) if the sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ of real numbers exist, for which: $\langle x_n \rangle \in \widetilde{x}$, $\langle y_n \rangle \in \widetilde{y}$ and

$$\{n \in \mathbf{N} | x_n < y_n\} \in \mathbf{\Phi ix}.$$

Theorem 2.10: For all Q-numbers $\widetilde{x}$, $\widetilde{y}$, $\widetilde{z}$: [5]

$\mathbf{\Omega 1}$: there is not that $\widetilde{x} < \widetilde{x}$;

$\mathbf{\Omega 2}$: if $\widetilde{x} < \widetilde{y}$ and $\widetilde{y} < \widetilde{z}$ then $\widetilde{x} < \widetilde{z}$;

$\mathbf{\Omega 4}$: if $\widetilde{x} < \widetilde{y}$ then $(\widetilde{x} + \widetilde{z}) < (\widetilde{y} + \widetilde{z})$;

$\mathbf{\Omega 5}$: if $0 < \widetilde{z}$ and $\widetilde{x} < \widetilde{y}$, then $(\widetilde{x} \cdot \widetilde{z}) < (\widetilde{y} \cdot \widetilde{z})$;

$\mathbf{\Omega 3'}$: if $\widetilde{x} < \widetilde{y}$ then there is not, that $\widetilde{y} < \widetilde{x}$ or $\widetilde{x} = \widetilde{y}$ and vice versa;

$\mathbf{\Omega 3''}$: for all standard Q-numbers x, y, z: $x < y$ or $y < x$ or $x = y$.

Theorem 2.11: $\mathbf{\Phi 8}$: If $0 < |\widetilde{x}|$ then the Q-number $\widetilde{y}$ exists, for which $(\widetilde{x} \cdot \widetilde{y}) = 1$.

Thus, Q-numbers are fulfilled to all properties of real numbers, except $\Omega 3$ [6]. The property $\Omega 3$ is accomplished by some weak meaning ($\Omega 3$' and $\Omega 3$").

Definition 2.11: The Q-number $\widetilde{x}$ is *the infinitesimal Q-number* if the sequence of real numbers $\langle x_n \rangle$ exists, for which: $\langle x_n \rangle \in \widetilde{x}$ and for all positive real numbers ε:

$$\{n \in \mathbf{N} | |x_n| < \varepsilon\} \in \mathbf{\Phi ix}.$$

Let the set of all infinitesimal Q-numbers be denoted as I.

Definition 2.12: The Q-numbers $\widetilde{x}$ and $\widetilde{y}$ are t*he infinite closed Q-numbers* (denote: $\widetilde{x} \approx \widetilde{y}$) if $|\widetilde{x} - \widetilde{y}| = 0$ or $|\widetilde{x} - \widetilde{y}|$ is infinitesimal.

Definition 2.13: The Q-number $\widetilde{x}$ is *the infinite Q-number* if the sequence $\langle r_n \rangle$ of real numbers exists, for which $\langle r_n \rangle \in \widetilde{x}$ and for every natural number m:

$$\{n \in \mathbf{N} | m < r_n\} \in \mathbf{\Phi ix}.$$

3 THE CLASSICAL LOGIC.

Definition 3.1: The sentence $\ll \Theta \gg$ is *the true sentence* if and only if Θ [1].

[1]Perhaps, the definition of the truth sentence belongs to A.Tarsky.

For example: the sentence $\ll$it rains$\gg$ is the true sentence if and only if it rains.

Definition 3.2: The sentence $\ll \Theta \gg$ is *the false sentence* if and only if it is not that Θ.

Definition 3.3: The sentences A and B are *equal* ($A = B$) if A is true if and only if B is true.

Hereinafter we use the usual notions of the classical propositional logic [7].

Definition 3.4: The sentence C is *the conjunction* of the sentences A and B ($C = (A \wedge B)$) , if C is true if and only if A is true and B is true.

Definition 3.5: The sentence C is *the negation* of the sentence A ($C = \overline{A}$), if C is true if and only if A is false.

Theorem 3.1:

1) $(A \wedge A) = A$;

2) $(A \wedge B) = (B \wedge A)$;

3) $(A \wedge (B \wedge C)) = ((A \wedge B) \wedge C)$;

4) if T is the true sentence then for every sentence A: $(A \wedge T) = A$.

Definition 3.6: Each function $\mathfrak{g}$, which has got the domain in the set of the sentences, has got the range of values on the two-elements set $\{0; 1\}$, is *the Boolean function* if:

1) for every sentence A: $\mathfrak{g}\left(\overline{A}\right) = 1 - \mathfrak{g}(A)$;

2) for all sentences A and B: $\mathfrak{g}(A \wedge B) = \mathfrak{g}(A) \cdot \mathfrak{g}(B)$.

Definition 3.7: The set $\Im$ of the sentences is *the basic set* if for every element A of this set the Boolean functions $\mathfrak{g}_1$ and $\mathfrak{g}_2$ exist, for which the following conditions fulfill:

1) $\mathfrak{g}_1(A) \neq \mathfrak{g}_2(A)$;

2) for each element B of $\Im$: if $B \neq A$ then $\mathfrak{g}_1(B) = \mathfrak{g}_2(B)$.

Definition 3.8: The set $[\Im]$ of the sentences is *the propositional closure* of the set $\Im$ if the following conditions fulfill:

1) if $A \in \Im$ then $\overline{A} \in \Im$;

2) if $A \in \Im$ and $B \in \Im$ then $(A \wedge B) \in \Im$.

In the following text the elements of $[\Im]$ are denoted as *the* $\Im$*-sentences*.

Definition 3.9: The $\Im$-sentence A is *the tautology* if for all Boolean functions $\mathfrak{g}$:

$$\mathfrak{g}(A) = 1.$$

Definition 3.10: *The disjunction* and *the implication* are defined by the usual way:

$$(A \vee B) = \overline{(\overline{A} \wedge \overline{B})},$$
$$(A \Rightarrow B) = \overline{(A \wedge \overline{B})}.$$

By this definition and the Definitions 3.4 and 3.5:

$(A \vee B)$ is the false sentence if and if, only, A is the false sentence and B is the false sentence.

$(A \Rightarrow B)$ is the false sentence if and if, only, A is the true sentence and B is the false sentence.

Definition 3.11: A $\Im$-sentence is *a propositional axiom* [8] if this sequence has got one some amongst the following forms:

A1. $(A \Rightarrow (B \Rightarrow A))$;

A2. $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$;

A3. $\left(\left(\overline{B} \Rightarrow \overline{A}\right) \Rightarrow \left(\left(\overline{B} \Rightarrow A\right) \Rightarrow B\right)\right)$.

Definition 3.12:

The sentence B is obtained from the sentences $(A \Rightarrow B)$ and A by the logic rule "*modus ponens*".

Definition 3.13: [9] The array $A_1, A_2, \ldots, A_n$ of the sentences is *the propositional deduction* of the sentence A from the hypothesis list Γ (denote: $\Gamma \vdash A$), if $A_n = A$ and for all numbers l $(1 \leq l \leq n)$: A_l is either the propositional axiom or A_l is obtained from some sentences A_{l-k} and A_{l-s} by the modus ponens or $A_l \in \Gamma$.

Definition 3.14: The sentence is t*he propositional proved sentence* if this sentence is the propositional axiom or this sentence is obtained from the propositional proved sentences by the modus ponens.

Hence, if A is the propositional proved sentence then the propositional deduction

$$\vdash A$$

exists.

Theorem 3.2: [10] If the sentence A is the propositional proved sentence then for all Boolean function $\mathfrak{g}$: $\mathfrak{g}(A) = 1$.

Theorem 3.3: (The completeness Theorem). [11] All tautologies are the propositional proved sentences.

4 B-FUNCTIONS

Definition 4.1: Each function $\mathfrak{b}(x)$, which has got the domain in the sentences set, has got the range of values on the numeric axis segment $[0;1]$, is named as *the B-function* if for every sentences A and B the following condition fulfills:

$$\mathfrak{b}(A \wedge B) + \mathfrak{b}\left(A \wedge \overline{B}\right) = \mathfrak{b}(A).$$

Theorem 4.1: For each B-function $\mathfrak{b}$:

1) for every sentences A and B: $\mathfrak{b}(A \wedge B) \leq \mathfrak{b}(A)$;

2) for every sentence A: if T is the true sentence, then $\mathfrak{b}(A) + \mathfrak{b}\overline{A} = \mathfrak{b}(T)$

3) for every sentence A: if T is the true sentence, then $\mathfrak{b}(A) \leq \mathfrak{b}(T)$;

Therefore, if the sentence C exists, for which: $\mathfrak{b}(C) = 1$, and T is the true sentence, then

$$\mathfrak{b}(T) = 1. \tag{1}$$

Hence, in this case for every sentence A:

$$\mathfrak{b}(A) + \mathfrak{b}\left(\overline{A}\right) = 1. \tag{2}$$

Theorem 4.2: If the sentence D is the propositional proved sentence then for all B-functions $\mathfrak{b}$: $\mathfrak{b}(D) = 1$.

Theorem 4.3:

1) If for all Boolean functions $\mathfrak{g}$:

$$\mathfrak{g}(A) = 1$$

then for all B-functions $\mathfrak{b}$:

$$\mathfrak{b}(A) = 1.$$

2) If for all Boolean functions $\mathfrak{g}$:

$$\mathfrak{g}(A) = 0$$

then for all B-functions $\mathfrak{b}$:

$$\mathfrak{b}(A) = 0.$$

Theorem 4.4: All Boolean functions are the B-functions.

Hence, the B-function is the generalization of the logic Boolean function. Therefore, the B-function is the logic function, too.

Theorem 4.5:

$$\mathfrak{b}\left(A \vee B\right) = \mathfrak{b}\left(A\right) + \mathfrak{b}\left(B\right) - \mathfrak{b}\left(A \wedge B\right).$$

Definition 4.2: The sentences A and B are *the inconsistent sentences for the B-function* $\mathfrak{b}$ if

$$\mathfrak{b}\left(A \wedge B\right) = 0.$$

Theorem 4.6: If the sentences A and B are the inconsistent sentences for the B-function $\mathfrak{b}$ then

$$\mathfrak{b}\left(A \vee B\right) = \mathfrak{b}\left(A\right) + \mathfrak{b}\left(B\right).$$

Theorem 4.7: If $\mathfrak{b}\left(A \wedge B\right) = \mathfrak{b}\left(A\right) \cdot \mathfrak{b}\left(B\right)$ then $\mathfrak{b}\left(A \wedge \overline{B}\right) = \mathfrak{b}\left(A\right) \cdot \mathfrak{b}\left(\overline{B}\right)$.

Theorem 4.8: $\mathfrak{b}\left(A \wedge \overline{A} \wedge B\right) = 0$.

5 THE INDEPENDENT TESTS

Definition 5.1: Let $\mathfrak{st}(n)$ be a function, which has got the domain on the set of natural numbers and has got the range of values in the set of the $\Im$-sentences.

In this case, the $\Im$-sentence A is *the [st]-series of the range r with the V-number k* if A, r and k fulfill to some one amongst the following conditions:

1) $r = 1$ and $k = 1$, $A = \mathfrak{st}\left(1\right)$ or $k = 0$, $A = \overline{\mathfrak{st}\left(1\right)}$;

2) B is the [st]-series of the range $r - 1$ with the V-number $k - 1$ and

$$A = \left(B \wedge \mathfrak{st}\left(r\right)\right),$$

or B is the [st]-series of the range $r - 1$ with the V-number k and

$$A = \left(B \wedge \overline{\mathfrak{st}\left(r\right)}\right).$$

Let us denote the set of the [st]-series of the range r with the V-number k as $[\mathfrak{st}](r, k)$.

For example, if $\mathfrak{st}(n)$ is the sentence B_n then the sentences:
$\left(B_1 \wedge B_2 \wedge \overline{B_3}\right)$, $\left(B_1 \wedge \overline{B_2} \wedge B_3\right)$, $\left(\overline{B_1} \wedge B_2 \wedge B_3\right)$
are the elements of $[\mathfrak{st}](3,2)$, and $\left(B_1 \wedge B_2 \wedge \overline{B_3} \wedge B_4 \wedge \overline{B_5}\right) \in [\mathfrak{st}](5,3)$.

Definition 5.2: The function $\mathfrak{st}(n)$ is *independent* for the B-function $\mathfrak{b}$ if for A: if $A \in [\mathfrak{st}](r,r)$ then:

$$\mathfrak{b}(A) = \prod_{n=1}^{r} \mathfrak{b}(\mathfrak{st}(n)).$$

Definition 5.3:.Let $\mathfrak{st}(n)$ be a function, which has got the domain on the set of natural numbers and has got the range of values in the set of the $\Im$-sentences.

In this case the sentence A is *the [st]-disjunction of the range r with the V-number k* (denote: $\mathfrak{t}[\mathfrak{st}](r,k)$) if A is the disjunction of all elements of $[\mathfrak{st}](r,k)$.

For example, if $\mathfrak{st}(n)$ is the sentence C_n then:
$\left(\overline{C_1} \wedge \overline{C_2} \wedge \overline{C_3}\right) = \mathfrak{t}[\mathfrak{st}](3,0)$,
$\mathfrak{t}[\mathfrak{st}](3,1) = \left(\left(C_1 \wedge \overline{C_2} \wedge \overline{C_3}\right) \vee \left(\overline{C_1} \wedge C_2 \wedge \overline{C_3}\right) \vee \left(\overline{C_1} \wedge \overline{C_2} \wedge C_3\right)\right)$,
$\mathfrak{t}[\mathfrak{st}](3,2) = \left(\left(C_1 \wedge C_2 \wedge \overline{C_3}\right) \vee \left(\overline{C_1} \wedge C_2 \wedge C_3\right) \vee \left(C_1 \wedge \overline{C_2} \wedge C_3\right)\right)$,
$(C_1 \wedge C_2 \wedge C_3) = \mathfrak{t}[\mathfrak{st}](3,3)$.

Definition 5.4:

$\nu_r[st](A)$ is *the frequency of the sentence A in the [st]-series of r independent for the B-function $\mathfrak{b}$ tests* if

1) $\mathfrak{st}(n)$ is independent for the B-function $\mathfrak{b}$,
2) for all n: $\mathfrak{b}(\mathfrak{st}(n)) = \mathfrak{b}(A)$,
3) $\mathfrak{t}[\mathfrak{st}](r,k) =$"$\nu_r[st](A) = \frac{k}{r}$".

Theorem 5.1: (the J.Bernoulli formula [12]**)** If $\mathfrak{st}(n)$ is independent for the B-function $\mathfrak{b}$, the real number p exists, for which: for all n: $\mathfrak{b}(\mathfrak{st}(n)) = p$, then

$$\mathfrak{b}(\mathfrak{t}[\mathfrak{st}](r,k)) = \frac{r!}{k! \cdot (r-k)!} \cdot p^k \cdot (1-p)^{r-k}.$$

Definition 5.5: Let $\mathfrak{st}(n)$ be a function, which has got the domain on the set of the natural numbers and has got the range of values in the set of the $\Im$-sentences.

In this case the function $\mathfrak{T}[\mathfrak{st}](r,k,l)$, which has got the domain in the set of threes of the natural numbers and has got the range of values in the set of the $\Im$-sentences, is defined recursively as follows:

1) $\mathfrak{T}[\mathfrak{st}](r,k,k) = \mathfrak{t}[\mathfrak{st}](r,k)$,

2) $\mathfrak{T}[\mathfrak{st}](r,k,l+1) = (\mathfrak{T}[\mathfrak{st}](r,k,l) \vee \mathfrak{t}[\mathfrak{st}](r,l+1))$.

Definition 5.6: If a and b are a real numbers and $k-1 < a \leq k$ and $l \leq b < l+1$ then $\mathfrak{T}[\mathfrak{st}](r,a,b) = \mathfrak{T}[\mathfrak{st}](r,k,l)$.

Theorem 5.2: If $\nu_r\,[st]\,(A)$ is the frequency of the sentence A in the [st]-series of r independent for the B-function $\mathfrak{b}$ tests then

$$\mathfrak{T}[\mathfrak{st}](r,a,b) = \text{"}\frac{a}{r} \leq \nu_r\,[st]\,(A) \leq \frac{b}{r}\text{"}.$$

Theorem 5.3: If $\mathfrak{st}(n)$ is independent for the B-function $\mathfrak{b}$, the real number p exists, for which: for all n: $\mathfrak{b}\,(\mathfrak{st}\,(n)) = p$, then

$$\mathfrak{b}\,(\mathfrak{T}[\mathfrak{st}](r,a,b)) = \sum_{a\leq k\leq b} \frac{r!}{k!\cdot(r-k)!}\cdot p^k\cdot(1-p)^{r-k}.$$

Theorem 5.4: If $\mathfrak{st}(n)$ is independent for the B-function $\mathfrak{b}$, the real number p exists, for which: for all n: $\mathfrak{b}\,(\mathfrak{st}\,(n)) = p$, then for every positive real number ε:

$$\mathfrak{b}\,(\mathfrak{T}[\mathfrak{st}](r, r\cdot(p-\varepsilon), r\cdot(p+\varepsilon))) \geq 1 - \frac{p\cdot(1-p)}{r\cdot\varepsilon^2}.$$

6 THE PROBABILITY FUNCTION

Definition 6.1: The sequences of the sentences $\langle A_n\rangle$ and $\langle B_n\rangle$ are *Q-equivalent* (denote: $\langle A_n\rangle \sim \langle B_n\rangle$) if

$$\{n \in \mathbf{N} | A_n = B_n\} \in \mathbf{\Phi ix}.$$

Definition 6.2: *The Q-sentence* is the set of the Q-equivalent sequences of the sentences, i.e. if $\widetilde{A}$ is the Q-number and $\mathbf{B} \in \widetilde{A}$ and $\mathbf{C} \in \widetilde{A}$, then $\mathbf{B} \sim \mathbf{C}$; and if $\mathbf{B} \in \widetilde{A}$ and $\mathbf{B} \sim \mathbf{C}$ then $\mathbf{C} \in \widetilde{A}$.

Definition 6.3: The Q-sentence $\widetilde{A}$ is *the standard Q-sentence* A if A is some sentence and sequence $\langle B_n\rangle$ exists, for which: $\langle B_n\rangle \in \widetilde{A}$ and

$$\{n \in \mathbf{N} | B_n = A\} \in \mathbf{\Phi ix}.$$

Definition 6.4: The Q-sentences $\widetilde{A}$ and $\widetilde{B}$ are *the equal Q-sentences* (denote: $\widetilde{A} = \widetilde{B}$) if a $\widetilde{A} \subseteq \widetilde{B}$ and $\widetilde{B} \subseteq \widetilde{A}$.

Definition 6.5: The function $\widetilde{\mathfrak{f}}$,which has got the domain in the set of the Q-sentences, has got the range of values in the set of the Q-numbers, is *the Q-extension of the function* $\mathfrak{f}$, which has got the domain in the set of the sentences, has the range of values in the set of the real numbers, if the following condition is accomplished:

if $\langle B_n \rangle \in \widetilde{B}$ and $\widetilde{\mathfrak{f}}(\widetilde{B}) = \widetilde{x}$ then $\langle \mathfrak{f}(B_n) \rangle \in \widetilde{x}$.

Definition 6.6: The function $\widetilde{\mathfrak{T}}$, which has got the domain in $Q\mathbf{R} \times Q\mathbf{R} \times Q\mathbf{R}$, has got the range of values in the set of the Q-sentences, is *the Q-extension of the function* $\mathfrak{T}$, which has got the domain in $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$, has the range of values in the set of the sentences, if the following condition is fulfilled:

if $\langle x_n \rangle \in \widetilde{x}$, $\langle y_n \rangle \in \widetilde{y}$, $\langle z_n \rangle \in \widetilde{z}$, $\widetilde{u} = \widetilde{\mathfrak{T}}(\widetilde{x}, \widetilde{y}, \widetilde{z})$, then $\langle \mathfrak{T}(x_n, y_n, z_n) \rangle \in \widetilde{u}$.

Theorem 6.1: Let x, y, z be the standard Q-numbers, B be the standard Q-sentence, $\widetilde{\mathfrak{f}}$ be the Q-extension of the function $\mathfrak{f}$, which has got the domain in the set of the sentences and has got the range of values in the set of the real numbers.

Let $\widetilde{\mathfrak{T}}$ be the Q-extension of the function $\mathfrak{T}$, which has got the domain in $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$ and has got the range of values in the set of the sentences.

In this case:

1) if $\widetilde{\mathfrak{f}}(B) = x$ then $\mathfrak{f}(B) = x$ and vice versa;

2) if $\widetilde{\mathfrak{T}}(x, y, z) = \widetilde{B}$ then $\mathfrak{T}(x, y, z) = B$ and vice versa;

By this Theorem: if $\widetilde{\mathfrak{f}}$ and $\widetilde{\mathfrak{T}}$ are the Q-extensions of the functions $\mathfrak{f}$ and $\mathfrak{T}$ correspondingly, then the expressions of the type: "$\widetilde{\mathfrak{f}}$" and "$\widetilde{\mathfrak{T}}$" shall be denote as "$\mathfrak{f}$" and "$\mathfrak{T}$" correspondingly. And if $\widetilde{B}$ is the standard Q-sentence B then the expressions of the type: "$\widetilde{B}$" will be denote as "B".

Theorem 6.2: For all Q-numbers $\widetilde{\varepsilon}$ and $\widetilde{r}$, for all functions $\mathfrak{st}(n)$, independent for the B-function $\mathfrak{b}$:

if $\widetilde{\varepsilon} > 0$ and a real number p exists, for which: for all natural n: $\mathfrak{st}(n) = p$, then

$$\mathfrak{b}\left(\mathfrak{T}[\mathfrak{st}](\widetilde{r}, \widetilde{r} \cdot (p - \widetilde{\varepsilon}), \widetilde{r} \cdot (p + \widetilde{\varepsilon}))\right) \geq 1 - \frac{p \cdot (1-p)}{\widetilde{r} \cdot \widetilde{\varepsilon}^2}.$$

Theorem 6.3: If $\widetilde{r}$ is the infinite Q-number then for all real positive numbers ε, for all functions $\mathfrak{st}(n)$, independent for the B-function $\mathfrak{b}$:

if a real number p exists, for which: for all natural n: $\mathfrak{st}(n) = p$, then

$$\mathfrak{b}\left(\mathfrak{T}[\mathfrak{st}](\widetilde{r}, \widetilde{r} \cdot (p - \widetilde{\varepsilon}), \widetilde{r} \cdot (p + \widetilde{\varepsilon}))\right) \approx 1.$$

Definition 6.7: The sentence "Θ" is *the almost authentic sentence* if it is real, that Θ.

For example:

Certain raffle is kept one million raffle tickets. The Big prize falls to the single ticket of this raffle. All tickets are sold. You have got one ticket.

In this case, the sentence "You shall not win the Big prize" is the almost authentic sentence because it is real that you shall not win the Big prize. But the sentence "Someone will win the Big prize." is the true sentence by the Definition 3.1.

Hence, all true sentences are the almost authentic sentences, but not all almost authentic sentences are the true sentences.

Definition 6.8: A function $\mathfrak{P}$ is a P-function if $\mathfrak{P}$ is a Q-extension of a B-function and the following condition is fulfilled:

for all Q-sentences $\widetilde{A}$: if $\mathfrak{P}\left(\widetilde{A}\right) \approx 1$ then $\widetilde{A}$ is an almost authentic sentence.

Theorem 6.4: If $\widetilde{r}$ is the infinite Q-number, $\nu_{\widetilde{r}}[st](A)$ is the frequency of the sentence A in the [st]-series of $\widetilde{r}$ independent for any P-function $\mathfrak{P}$ tests, then it is real, that for each real positive number ε:

$$|\nu_{\widetilde{r}}[st](A) - \mathfrak{P}(A)| < \varepsilon.$$

Theorem 6.5: If $\widetilde{r}$ is the infinite Q-number, $\nu_{\widetilde{r}}[st](A)$ is the frequency of the sentence A in the [st]-series of $\widetilde{r}$ independent for a P-function $\mathfrak{P}$ tests, then it is real, that

$$\nu_{\widetilde{r}}[st](A) \approx \mathfrak{P}(A).$$

Therefore, **the function, defined by the Definition 6.8 has got the statistical meaning.** That is why I'm name such function as *the probability function.*

7 RESUME

The probability function is the extension of the logic B-function. Therefore, **the probability is some generalization of the classic propositional logic.**

8 APPENDIX I. Consistency

Let us define the propositional calculus like to ([7]), but the propositional forms shall be marked by the script greek letters.

Definition C1: A set $\Re$ of the propositional forms is *a U-world* if:

1) if $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Re$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \vdash \beta$ then $\beta \in \Re$,

2) for all propositional forms α: it is not that $(\alpha \& (\neg\alpha)) \in \Re$,

3) for every propositional form α: $\alpha \in \Re$ or $(\neg\alpha) \in \Re$.

Definition C2: The sequences of the propositional forms $\langle \alpha_n \rangle$ and $\langle \beta_n \rangle$ are *Q-equivalent* (denote: $\langle \alpha_n \rangle \sim \langle \beta_n \rangle$) if

$$\{n \in \mathbf{N} | \alpha_n \equiv \beta_n\} \in \mathbf{\Phi ix}.$$

Let us define the notions of *the Q-forms* and *the Q-extension of the functions* for the propositional forms like as in the Definitions 2.5, 6.2, 2.9, 6.5, 6.6.

Definition C3: The Q-form $\widetilde{\alpha}$ is *Q-real* in the U-world $\Re$ if the sequence $\langle \alpha_n \rangle$ of the propositional forms exists, for which: $\langle \alpha_n \rangle \in \widetilde{\alpha}$ and

$$\{n \in \mathbf{N} | \alpha_n \in \Re\} \in \mathbf{\Phi ix}.$$

Definition C4: The set $\widetilde{\Re}$ of the Q-forms is t*he Q-extension of the U-world* $\Re$ if $\widetilde{\Re}$ is the set of Q-forms $\widetilde{\alpha}$, which are Q-real in $\Re$.

Definition C5: The sequence $\left\langle \widetilde{\Re}_k \right\rangle$ of the Q-extensions is *the S-world.*

Definition C6: The Q-form $\widetilde{\alpha}$ is *S-real in the S-world* $\left\langle \widetilde{\Re}_k \right\rangle$ if

$$\left\{k \in \mathbf{N} | \widetilde{\alpha} \in \widetilde{\Re}_k \right\} \in \mathbf{\Phi ix}.$$

Definition C7: The set $\mathbf{A}$ ($\mathbf{A} \subseteq \mathbf{N}$) is *the regular set* if for every real positive number ε the natural number n_0 exists, for which: for all natural numbers n and m, which are more or equal to n_0:

$$|w_n(\mathbf{A}) - w_m(\mathbf{A})| < \varepsilon.$$

Theorem C1: If $\mathbf{A}$ is the regular set and for all real positive ε:

$$\{k \in \mathbf{N} | w_k(\mathbf{A}) < \varepsilon\} \in \mathbf{\Phi ix}.$$

then

$$\lim_{k \to \infty} w_k(\mathbf{A}) = 0.$$

Proof of theTheorem C1: Let be

$$\lim_{k \to \infty} w_k(\mathbf{A}) \neq 0.$$

That is the real number ε_0 exists, for which: for every natural number n' the natural number n exists, for which:

$$n > n' \text{ and } w_n(\mathbf{A}) > \varepsilon_0.$$

Let δ_0 be some positive real number, for which: $\varepsilon_0 - \delta_0 > 0$. Because $\mathbf{A}$ is the regular set then for δ_0 the natural number n_0 exists, for which: for all natural numbers n and m, which are more or equal to n_0:

$$|w_m(\mathbf{A}) - w_n(\mathbf{A})| < \delta_0.$$

That is

$$w_m(\mathbf{A}) > w_n(\mathbf{A}) - \delta_0.$$

Since $w_n(\mathbf{A}) \geq \varepsilon_0$ then $w_m(\mathbf{A}) \geq \varepsilon_0 - \delta_0$.

Hence, the natural number n_0 exists, for which: for all natural numbers m: if $m \geq n_0$ then $w_m(\mathbf{A}) \geq \varepsilon_0 - \delta_0$.

Therefore,

$$\{m \in \mathbf{N} | w_m(\mathbf{A}) \geq \varepsilon_0 - \delta_0\} \in \mathbf{\Phi ix}.$$

and by this Theorem condition:

$$\{k \in \mathbf{N} | w_k(\mathbf{A}) < \varepsilon_0 - \delta_0\} \in \mathbf{\Phi ix}.$$

Hence,

$$\{k \in \mathbf{N} | \varepsilon_0 - \delta_0 < \varepsilon_0 - \delta_0\} \in \mathbf{\Phi ix}.$$

That is $\emptyset \notin \mathbf{\Phi ix}$. It is the contradiction for the Theorem 2.2.

Definition C8: Let $\left\langle \widetilde{\Re}_k \right\rangle$ be a S-world.

In this case the function $\mathfrak{W}(\widetilde{\beta})$, which has got the domain in the set of the Q-forms, has got the range of values in $Q\mathbf{R}$, is defined as the following:

If $\mathfrak{W}(\widetilde{\beta}) = \widetilde{p}$ then the sequence $\langle p_n \rangle$ of the real numbers exists, for which: $\langle p_n \rangle \in \widetilde{p}$ and

$$p_n = w_n \left(\left\{ k \in \mathbf{N} | \widetilde{\beta} \in \widetilde{\Re}_k \right\} \right).$$

Theorem C2: If $\left\{ k \in \mathbf{N} | \widetilde{\beta} \in \widetilde{\Re}_k \right\}$ is the regular set and $\mathfrak{W}(\widetilde{\beta}) \approx 1$ then $\widetilde{\beta}$ is S-resl in $\left\langle \widetilde{\Re}_k \right\rangle$.

Proof of the Theorem C2: Since $\mathfrak{W}(\widetilde{\beta}) \approx 1$ then by Definitions.2.12 and 2.11: for all positive real ε:

$$\left\{ n \in \mathbf{N} | w_n \left(\left\{ k \in \mathbf{N} | \widetilde{\beta} \in \widetilde{\Re}_k \right\} \right) > 1 - \varepsilon \right\} \in \mathbf{\Phi ix}.$$

Hence, by the point 3 of the Theorem 2.1: for all positive real ε:

$$\left\{ n \in \mathbf{N} | \left(\mathbf{N} - w_n \left(\left\{ k \in \mathbf{N} | \widetilde{\beta} \in \widetilde{\Re}_k \right\} \right) \right) < \varepsilon \right\} \in \mathbf{\Phi ix}.$$

Therefore, by the Theorem C1:

$$\lim_{n \to \infty} \left(\mathbf{N} - w_n \left(\left\{ k \in \mathbf{N} | \widetilde{\beta} \in \widetilde{\Re}_k \right\} \right) \right) = 0.$$

That is:

$$\lim_{n \to \infty} w_n \left(\left\{ k \in \mathbf{N} | \widetilde{\beta} \in \widetilde{\Re}_k \right\} \right) = 1.$$

Hence, by Definition.2.3:

$$\left\{k \in \mathbf{N} | \widetilde{\beta} \in \widetilde{\Re}_k\right\} \in \mathbf{\Phi ix}.$$

And by Definition C6: $\widetilde{\beta}$ is S-real in $\left\langle \widetilde{\Re}_k \right\rangle$.

Theorem C3: The P-function exists.

Proof of the Theorem C3: By the Theorems C2 and 2.1: $\mathfrak{W}(\widetilde{\beta})$ is the P-function in $\left\langle \widetilde{\Re}_k \right\rangle$.

9 APPENDIX II. Proofs

This Appendix contains proofs of the Theorems:

Proof of the Theorem 2.1: This is obvious.

Proof of the Theorem 2.2: From the point 3 of Theorem 2.1:

$$\lim_{n\to\infty} \varpi_n(\mathbf{N} - \mathbf{B}) = 0.$$

From the point 4 of Theorem 2.1:

$$\varpi_n(\mathbf{A} \cap (\mathbf{N} - \mathbf{B})) \leq \varpi_n(\mathbf{N} - \mathbf{B}).$$

Hence,

$$\lim_{n\to\infty} \varpi_n\left(\mathbf{A} \cap (\mathbf{N} - \mathbf{B})\right) = 0.$$

Hence,

$$\lim_{n\to\infty} \varpi_n\left(\mathbf{A} \cap \mathbf{B}\right) = \lim_{n\to\infty} \varpi_n(\mathbf{A}).$$

Proof of the Theorem 2.3: By Definition 2.4 from the Theorem 2.2 is obvious.

Proof of the Theorem 2.4: Let us denote:

if $k = 1$ or $k = 2$ or $k = 3$ then

$$\mathbf{A}_k = \left\{n \in \mathbf{N} | y_{k,n} = z_{k,n}\right\}.$$

In this case by Definition 2.4 for all k:

$$\mathbf{A}_k \in \mathbf{\Phi ix}.$$

Because

$$(\mathbf{A}_1 \cap \mathbf{A}_2 \cap \mathbf{A}_3) \subseteq \{n \in \mathbf{N} | \mathfrak{f}(y_{1,n}, y_{2,n}, y_{3,n}) = \mathfrak{f}(z_{1,n}, z_{2,n}, z_{3,n})\},$$

then by Theorem 2.2:

$$\{n \in \mathbf{N} | \mathfrak{f}(y_{1,n}, y_{2,n}, y_{3,n}) = \mathfrak{f}(z_{1,n}, z_{2,n}, z_{3,n})\} \in \mathbf{\Phi ix}.$$

Proof of the Theorem 2.5: If $\langle r_n \rangle \in \widetilde{a}$, $\langle s_n \rangle \in \widetilde{b}$, $\langle u_n \rangle \in \widetilde{c}$, $\langle \mathfrak{t}_n \rangle \in \widetilde{d}$ then by Definition 2.6:

$$\begin{array}{l} \{n \in \mathbf{N} | r_n = a\} \in \mathbf{\Phi ix}, \\ \{n \in \mathbf{N} | s_n = b\} \in \mathbf{\Phi ix}, \\ \{n \in \mathbf{N} | u_n = c\} \in \mathbf{\Phi ix}, \\ \{n \in \mathbf{N} | t_n = d\} \in \mathbf{\Phi ix}. \end{array}$$

1) Let $d = \mathfrak{f}(a, b, c)$.
In this case by Theorem 2.2:

$$\{n \in \mathbf{N} | t_n = \mathfrak{f}(r_n, s_n, u_n)\} \in \mathbf{\Phi ix}.$$

Hence, by Definition 2.4:

$$\langle t_n \rangle \sim \langle \mathfrak{f}(r_n, s_n, u_n) \rangle.$$

Therefore by Definition 2.5:

$$\langle \mathfrak{f}(r_n, s_n, u_n) \rangle \in \widetilde{d}.$$

Hence, by Definition 2.9:

$$\widetilde{d} = \widetilde{\mathfrak{f}}(\widetilde{a}, \widetilde{b}, \widetilde{c}).$$

2) Let $\widetilde{d} = \widetilde{\mathfrak{f}}(\widetilde{a}, \widetilde{b}, \widetilde{c})$.
In this case by Definition 2.9:

$$\langle \mathfrak{f}(r_n, s_n, u_n) \rangle \in \widetilde{d}.$$

Hence, by Definition 2.5:

$$\langle t_n \rangle \sim \langle \mathfrak{f}(r_n, s_n, u_n) \rangle .$$

Therefore, by Definition 2.4:

$$\{n \in \mathbf{N} | t_n = \mathfrak{f}(r_n, s_n, u_n)\} \in \mathbf{\Phi ix}.$$

Hence, by the Theorem 2.2:

$$\{n \in \mathbf{N} | t_n = \mathfrak{f}(r_n, s_n, u_n), r_n = a, s_n = b, u_n = c, t_n = d\} \in \mathbf{\Phi ix}.$$

Hence, since this set does not empty, then

$$d = \mathfrak{f}(a, b, c).$$

Proof of the Theorem 2.6: If $\langle x_n \rangle \in \widetilde{x}$, $\langle y_n \rangle \in \widetilde{y}$, $\langle z_n \rangle \in \widetilde{z}$, $\widetilde{u} = \varphi(\widetilde{x}, \widetilde{y}, \widetilde{z})$, then by Definition 2.9: $\langle \varphi(x_n, y_n, z_n) \rangle \in \widetilde{u}$.

Because $\varphi(x_n, y_n, z_n) = \psi(x_n, y_n, z_n)$ then $\langle \psi(x_n, y_n, z_n) \rangle \in \widetilde{u}$.

If $\widetilde{v} = \psi(\widetilde{x}, \widetilde{y}, \widetilde{z})$ then by Definition 2.9: $\langle \psi(x_n, y_n, z_n) \rangle \in \widetilde{v}$, too.

Therefore, for all sequences $\langle t_n \rangle$ of real numbers: if $\langle t_n \rangle \in \widetilde{u}$ then by Definition 2.5: $\langle t_n \rangle \sim \langle \psi(x_n, y_n, z_n) \rangle$.

Hence, $\langle t_n \rangle \in \widetilde{v}$; and if $\langle t_n \rangle \in \widetilde{v}$ then $\langle t_n \rangle \sim \langle \varphi(x_n, y_n, z_n) \rangle$; hence, $\langle t_n \rangle \in \widetilde{u}$.

Therefore, $\widetilde{u} = \widetilde{v}$.

Proof of the Theorem 2.7: Let $\langle w_n \rangle \in \widetilde{w}$, $\mathfrak{f}(\widetilde{x}, \widetilde{w}) = \widetilde{u}$, $\langle x_n \rangle \in \widetilde{x}$, $\langle y_n \rangle \in \widetilde{y}$, $\langle z_n \rangle \in \widetilde{z}$, $\varphi(\widetilde{y}, \widetilde{z}) = \widetilde{w}$, $\psi(\widetilde{x}, \widetilde{y}, \widetilde{z}) = \widetilde{v}$.

By the condition of this Theorem: $\mathfrak{f}(x_n, \varphi(y_n, z_n)) = \psi(x_n, y_n, z_n)$.

By Definition 2.9: $\langle \psi(x_n, y_n, z_n) \rangle \in \widetilde{v}$, $\langle \varphi(x_n, y_n) \rangle \in \widetilde{w}$, $\langle \mathfrak{f}(x_n, w_n) \rangle \in \widetilde{u}$.

For all sequences $\langle t_n \rangle$ of real numbers:

1) If $\langle t_n \rangle \in \widetilde{v}$ then by Definition 2.5: $\langle t_n \rangle \sim \langle \psi(x_n, y_n, z_n) \rangle$.

Hence $\langle t_n \rangle \sim \langle \mathfrak{f}(x_n, \varphi(y_n, z_n)) \rangle$.

Therefore, by Definition 2.4:

$$\{n \in \mathbf{N} | t_n = \mathfrak{f}(x_n, \varphi\left(y_n, z_n\right))\} \in \mathbf{\Phi ix}$$

and

$$\{n \in \mathbf{N} | w_n = \varphi\left(y_n, z_n\right)\} \in \mathbf{\Phi ix}.$$

Hence, by Theorem 2.2:

$$\{n \in \mathbf{N} | t_n = \mathfrak{f}(x_n, w_n)\} \in \mathbf{\Phi ix}.$$

Hence, by Definition 2.4:

$$\langle t_n \rangle \sim \langle \mathfrak{f}(x_n, w_n) \rangle .$$

Therefore, by Definition 2.5: $\langle t_n \rangle \in \widetilde{u}$.
2) If $\langle t_n \rangle \in \widetilde{u}$ then by Definition 2.5: $\langle t_n \rangle \sim \langle \mathfrak{f}(x_n, w_n) \rangle$.
Because $\langle w_n \rangle \sim \langle \varphi(y_n, z_n) \rangle$ then by Definition 2.4:

$$\{n \in \mathbf{N} | t_n = \mathfrak{f}(x_n, w_n)\} \in \mathbf{\Phi ix},$$

$$\{n \in \mathbf{N} | w_n = \varphi\left(y_n, z_n\right)\} \in \mathbf{\Phi ix}.$$

Therefore, by Theorem 2.2:

$$\{n \in \mathbf{N} | t_n = \mathfrak{f}(x_n, \varphi\left(y_n, z_n\right))\} \in \mathbf{\Phi ix}.$$

Hence, by Definition 2.4:

$$\langle t_n \rangle \sim \langle \mathfrak{f}(x_n, \varphi(y_n, z_n)) \rangle .$$

Therefore,

$$\langle t_n \rangle \sim \langle \psi(x_n, y_n, z_n) \rangle .$$

Hence, by Definition 2.5: $\langle t_n \rangle \in \widetilde{v}$.
From above and from 1) by Definition 2.7: $\widetilde{u} = \widetilde{v}$.

Proof of the Theorem 2.8: If $\langle x_n \rangle \in \widetilde{x}$ then $\widetilde{y}$ is the Q-number, which contains $\langle -x_n \rangle$.

Proof of the Theorem 2.9: is obvious from Definition 2.6 and Definition 2.7.

Proof of the Theorem 2.10: is obvious from Definition 2.10 by the Theorem 2.2.

Proof of the Theorem 2.11: If $\langle x_n \rangle \in \widetilde{x}$ then by Definition 2.10: if

$$\mathbf{A} = \{n \in \mathbf{N} | 0 < |x_n|\}$$

then $\mathbf{A} \in \mathbf{\Phi ix}$.

In this case: if for the sequence $\langle y_n \rangle$: if $n \in \mathbf{A}$ then $y_n = 1/x_n$

- then

$$\{n \in \mathbf{N} | x_n \cdot y_n = 1\} \in \mathbf{\Phi ix}.$$

Proof of the Theorem 3.1: It is obviously.

Proof of the Theorem 3.2: in [10].

Proof of the Theorem 3.3: in [11].

Proof of the Theorem 4.1:

1) It is obviously.

2) From the points 4 and 2 of the Theorem 3.1: $\mathfrak{b}\left(T \wedge A\right) + \mathfrak{b}\left(T \wedge \overline{A}\right) = \mathfrak{b}\left(A\right) + \mathfrak{b}\left(\overline{A}\right)$.

3) It is obviously.

Proof of the Theorem 4.2:

If D is A1 then by Definition.3.10:

$$\mathfrak{b}\left(D\right) = \mathfrak{b}\left(\overline{\left(A \wedge \overline{\overline{\left(B \wedge \overline{A}\right)}}\right)}\right).$$

By (2):

$$\mathfrak{b}\left(D\right) = 1 - \mathfrak{b}\left(A \wedge \overline{\overline{\left(B \wedge \overline{A}\right)}}\right).$$

By the Definition 4.1 and the Theorem 3.1:

$$\mathfrak{b}\left(D\right) = 1 - \mathfrak{b}\left(A\right) + \mathfrak{b}\left(A \wedge \overline{\left(B \wedge \overline{A}\right)}\right),$$
$$\mathfrak{b}\left(D\right) = 1 - \mathfrak{b}\left(A\right) + \mathfrak{b}\left(A\right) - \mathfrak{b}\left(A \wedge \left(B \wedge \overline{A}\right)\right),$$
$$\mathfrak{b}\left(D\right) = 1 - \mathfrak{b}\left(\left(A \wedge B\right) \wedge \overline{A}\right),$$
$$\mathfrak{b}\left(D\right) = 1 - \mathfrak{b}\left(A \wedge B\right) + \mathfrak{b}\left(\left(A \wedge B\right) \wedge A\right),$$
$$\mathfrak{b}\left(D\right) = 1 - \mathfrak{b}\left(A \wedge B\right) + \mathfrak{b}\left(\left(A \wedge A\right) \wedge B\right),$$
$$\mathfrak{b}\left(D\right) = 1 - \mathfrak{b}\left(A \wedge B\right) + \mathfrak{b}\left(A \wedge B\right).$$

The proof is similar for the rest propositional axioms .
Let for all B-function $\mathfrak{b}$: $\mathfrak{b}(A) = 1$ and $\mathfrak{b}(A \Rightarrow D) = 1$.
By Definition.3.10:

$$\mathfrak{b}\left(A \Rightarrow D\right) = \mathfrak{b}\left(\overline{A \wedge \overline{D}}\right).$$

By (2):

$$\mathfrak{b}\left(A \Rightarrow D\right) = 1 - \mathfrak{b}\left(A \wedge \overline{D}\right).$$

Hence,

$$\mathfrak{b}\left(A \wedge \overline{D}\right) = 0.$$

By Definition.4.1:

$$\mathfrak{b}\left(A \wedge \overline{D}\right) = \mathfrak{b}\left(A\right) - \mathfrak{b}\left(A \wedge D\right).$$

Hence,

$$\mathfrak{b}\left(A \wedge D\right) = \mathfrak{b}\left(A\right) = 1.$$

By Definition.4.1 and the Theorem 3.1:

$$\mathfrak{b}\left(A \wedge D\right) = \mathfrak{b}\left(D\right) - \mathfrak{b}\left(D \wedge \overline{A}\right) = 1.$$

Therefore, for all B-function $\mathfrak{b}$:

$$\mathfrak{b}\left(D\right) = 1.$$

Proof of the Theorem 4.3:
1) This just follows from the preceding Theorem and from the Theorem 3.3.
2) If for all Boolean functions $\mathfrak{g}$: $\mathfrak{g}\left(A\right) = 0$, then by the Definition 3.6: $\mathfrak{g}\left(\overline{A}\right) = 1$. Hence, by the point 1 of this Theorem: for all B-function $\mathfrak{b}$: $\mathfrak{b}\left(\overline{A}\right) = 1$. By (2): $\mathfrak{b}\left(A\right) = 0$.
Proof of the Theorem 4.4: By Definition 3.6: for all Boolean functions $\mathfrak{g}$:

$$\mathfrak{g}\left(A \wedge B\right) + \mathfrak{g}\left(A \wedge \overline{B}\right) = \mathfrak{g}\left(A\right) \cdot \mathfrak{g}\left(B\right) + \mathfrak{g}\left(A\right) \cdot \left(1 - \mathfrak{g}\left(B\right)\right) = \mathfrak{g}\left(A\right).$$

Proof of the Theorem 4.5: By the Definition 3.10 and (2):

$$\mathfrak{b}\left(A \vee B\right) = 1 - \mathfrak{b}\left(\overline{A} \wedge \overline{B}\right).$$

By Definition 4.1:

$$\mathfrak{b}\left(A \vee B\right) = 1 - \mathfrak{b}\left(\overline{A}\right) + \mathfrak{b}\left(\overline{A} \wedge B\right) = \mathfrak{b}\left(A\right) + \mathfrak{b}\left(B\right) - \mathfrak{b}\left(A \wedge B\right).$$

Proof of the Theorem 4.6: This just follows from the preceding Theorem and Definition.

Proof of the Theorem 4.7: By the Definition 4.1:

$$\mathfrak{b}\left(A \wedge \overline{B}\right) = \mathfrak{b}\left(A\right) - \mathfrak{b}\left(A \wedge B\right).$$

Hence,

$$\mathfrak{b}\left(A \wedge \overline{B}\right) = \mathfrak{b}\left(A\right) - \mathfrak{b}\left(A\right) \cdot \mathfrak{b}\left(B\right) = \mathfrak{b}\left(A\right) \cdot \left(1 - \mathfrak{b}\left(B\right)\right).$$

Hence, by (2):

$$\mathfrak{b}\left(A \wedge \overline{B}\right) = \mathfrak{b}\left(A\right) \cdot \mathfrak{b}\left(\overline{B}\right).$$

Proof of the Theorem 4.8: By the Definition 4.1 and by the points 2 and 3 of the Theorem 3.1:

$$\mathfrak{b}\left(A \wedge \overline{A} \wedge B\right) = \mathfrak{b}\left(A \wedge B\right) - \mathfrak{b}\left(A \wedge A \wedge B\right),$$

hence, by the point 1 of the Theorem 3.1:

$$\mathfrak{b}\left(A \wedge \overline{A} \wedge B\right) = \mathfrak{b}\left(A \wedge B\right) - \mathfrak{b}\left(A \wedge B\right).$$

Proof of the Theorem 5.1: By the Definition 5.2 and the Theorem 4.7: if $B \in [\mathfrak{st}]\left(r, k\right)$ then:

$$\mathfrak{b}\left(B\right) = p^k \cdot \left(1 - p\right)^{r-k}.$$

Since $[\mathfrak{st}]\left(r, k\right)$ contains $\frac{r!}{k! \cdot (r-k)!}$ elements then by the Theorems 4.8 and 4.6 this Theorem is fulfilled.

Proof of the Theorem 5.2: By the Definition 5.6: the natural numbers r and k exist, for which: $k-1 < a \leq k$ and $l \leq b < l+1$.

The induction on l:

1. Let $l = k$.

In this case by the Definition 5.4:

$$\mathfrak{T}[\mathfrak{st}](r,k,k) = \mathfrak{t}[\mathfrak{st}](r,k) = \text{"}\nu_r\left[st\right](A) = \frac{k}{r}\text{"}.$$

2. Let n be any natural number.

The inductive supposition: Let

$$\mathfrak{T}[\mathfrak{st}](r,k,k+n) = \text{"}\frac{k}{r} \leq \nu_r\left[st\right](A) \leq \frac{k+n}{r}\text{"}.$$

By the Definition 5.5:

$$\mathfrak{T}[\mathfrak{st}](r,k,k+n+1) = (\mathfrak{T}[\mathfrak{st}](r,k,k+n) \vee \mathfrak{t}[\mathfrak{st}](r,k+n+1)).$$

By the inductive supposition and by the Definition 5.4:

$$\mathfrak{T}[\mathfrak{st}](r,k,k+n+1) =$$

$$= (\text{"}\frac{k}{r} \leq \nu_r\left[st\right](A) \leq \frac{k+n}{r}\text{"} \vee \text{"}\nu_r\left[st\right](A) = \frac{k+n+1}{r}\text{"}).$$

Hence, by the Definition 3.10:

$$\mathfrak{T}[\mathfrak{st}](r,k,k+n+1) = \text{"}\frac{k}{r} \leq \nu_r\left[st\right](A) \leq \frac{k+n+1}{r}\text{"}.$$

Proof of the Theorem 5.3: This is the consequence from the Theorem 5.1 by the Theorem 4.6.

Proof of the Theorem 5.4: Because

$$\sum_{k=0}^{r}(k-r\cdot p)^2\cdot\frac{r!}{k!\cdot(r-k)!}\cdot p^k\cdot(1-p)^{r-k} = r\cdot p\cdot(1-p)$$

then if

$$J = \{k\in\mathbf{N}|0\leq k\leq r\cdot(p-\varepsilon)\}\cap\{k\in\mathbf{N}|r\cdot(p+\varepsilon)\leq k\leq r\}$$

then

$$\sum_{k\in J}\frac{r!}{k!\cdot(r-k)!}\cdot p^{k}\cdot(1-p)^{r-k}\leq\frac{p\cdot(1-p)}{r\cdot\varepsilon^{2}}.$$

Hence, by (2) this Theorem is fulfilled.

Proof of the Theorem 6.1: is in common with the Proof of the Theorem 2.5.

Proof of the Theorem 6.2: This just follows from the preceding Theorem and from the Theorem 5.4 and the Definition 2.10.

Proof of the Theorem 6.3: By the Definition.2.13: the sequence $\langle r_n\rangle$ of real numbers exists, for which $\langle r_n\rangle\in\widetilde{r}$ and for every natural number m:

$$\{n\in\mathbf{N}|m<r_n\}\in\mathbf{\Phi ix}.$$

Hence, for all real positive numbers δ:

$$\left\{n\in\mathbf{N}|\frac{p\cdot(1-p)}{\widetilde{r}\cdot\widetilde{\varepsilon}^{2}}<\delta\right\}\in\mathbf{\Phi ix}.$$

Therefore, by the Definitions.2.11: $\frac{p\cdot(1-p)}{\widetilde{r}\cdot\widetilde{\varepsilon}^{2}}$ is the infinitesimal. Because by the Definition.4.1 $\mathfrak{b}\leq 1$ then by the Definition.2.12: from the Theorem 6.2 this Theorem is fulfilled.

Proof of the Theorem 6.4: By the Theorem 6.3:

$$\mathfrak{P}\left(\mathfrak{T}[\mathfrak{st}](\widetilde{r},\widetilde{r}\cdot(\mathfrak{P}(A)-\widetilde{\varepsilon}),\widetilde{r}\cdot(\mathfrak{P}(A)+\widetilde{\varepsilon}))\right)\approx 1.$$

By the Definitions 6.8: The sentence

$$\mathfrak{T}[\mathfrak{st}](\widetilde{r},\widetilde{r}\cdot(\mathfrak{P}(A)-\widetilde{\varepsilon}),\widetilde{r}\cdot(\mathfrak{P}(A)+\widetilde{\varepsilon}))$$

is the almost authentic sentence.

By the Theorem 5.2:

$$\mathfrak{T}[\mathfrak{st}](\widetilde{r},\widetilde{r}\cdot(\mathfrak{P}(A)-\widetilde{\varepsilon}),\widetilde{r}\cdot(\mathfrak{P}(A)+\widetilde{\varepsilon}))=$$

$$=\text{''}\,(\mathfrak{P}(A)-\widetilde{\varepsilon})\leq\nu_{\widetilde{r}}[st](A)\leq(\mathfrak{P}(A)+\widetilde{\varepsilon})\,\text{''}.$$

Hence, by the Definition 6.7: It is real, that $(\mathfrak{P}(A) - \widetilde{\varepsilon}) \leq \nu_{\widetilde{r}}[st](A) \leq (\mathfrak{P}(A) + \widetilde{\varepsilon})$.

Proof of the Theorem 6.5: Because from the Theorem 6.4 it is real, that for each real positive number ε:

$$|\nu_{\widetilde{r}}[st](A) - \mathfrak{P}(A)| < \varepsilon,$$

then by the Definition 2.11: $|\nu_{\widetilde{r}}[st](A) - \mathfrak{P}(A)|$ is the infinitesimal Q-number. Hence, by the Definition 2.12 this Theorem is fulfilled.

References

[1] J.Bernoulli. On the Large Number Law. (Moscow, 1986)

[2] M.Davis. Applied Nonstandard Analysis. (Moscow, 1980), p.32

[3] Item

[4] Item, p.73

[5] Item, p.73

[6] Item, p.74

[7] Elliot Mendelson. Introduction to Mathematical Logic. (Moscow, 1971)

[8] Item, p.38

[9] Item, p.37

[10] Item, p.43

[11] Item, p.44

[12] J.Bernoulli. On the Large Number Law. (Moscow, 1986), p.57